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ON THE NEGATIVE PELL EQUATION $x^2 = 6y^2 - 50$

S. Vidhyalakshmi[1], M.A. Gopalan[2] & T. Mahalakshmi*[3]
[1,2]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India
[*3]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India

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ABSTRACT

The binary quadratic equation represented by the negative pellian $x^2 = 6y^2 - 50$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

KEYWORDS: binary quadratic, hyperbola, parabola, pell equation, integral solutions.

1. INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 - 1$, where $D > 0$ and square free, is known as negative pell equation. In general, the general form of negative pell equation is represented by $y^2 = Dx^2 - N$, $N > 0$, $D > 0$ and square free. It is known that negative pell equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions whereas $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. It is observed that the negative pell equation do not always have integer solutions. For negative pell equations with integer solutions, one may refer [1-11].

In this communication, yet another negative pell equation given by $y^2 = 7x^2 - 14$ is considered for its non-zero distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

2. METHOD OF ANALYSIS

The negative pell equation representing hyperbola under consideration is

$$x^2 = 6y^2 - 50 \tag{1}$$

whose smallest positive integer solution is $x_0 = 2, y_0 = 3$

To obtain the other solutions of (1), consider the pell equation

$$x^2 = 6y^2 + 1$$

whose solution is given by

$$\tilde{x}_n = \frac{f_n}{2}, \tilde{y}_n = \frac{g_n}{2\sqrt{6}}$$

where $f_n = \left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1}$

$$g_n = \left(5 + 2\sqrt{6}\right)^{n+1} - \left(5 - 2\sqrt{6}\right)^{n+1}$$

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Appling Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other solutions of (1) are given by

$$6x_{n+1} = 6f_n + 9\sqrt{6}g_n$$

$$6y_{n+1} = 9f_n + \sqrt{6}g_n \ , \ n = -1,0,1,2....$$

The recurrence relations satisfied by the solutions x and y are given by

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below

Table: 1 Examples

n	x_{n+1}	y_{n+1}
-1	2	3
0	46	19
1	458	187
2	4534	1851
3	44882	18323
4	444286	181379

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} values are even and y_{n+1} values are odd
2. Each of the following expressions is a nasty number:
   - $\frac{1}{25}(9x_{2n+3} - 57x_{2n+2} + 300)$
   - $\frac{1}{250}(9x_{2n+4} - 561x_{2n+2} + 3000)$
   - $\frac{1}{25}(108y_{2n+2} - 12x_{2n+2} + 300)$
   - $\frac{1}{125}(108y_{2n+3} - 276x_{2n+2} + 1500)$
   - $\frac{1}{1225}(108y_{2n+4} - 2748x_{2n+2} + 14700)$
   - $\frac{1}{25}(57x_{2n+4} - 561x_{2n+3} + 300)$
   - $\frac{1}{125}(684y_{2n+2} - 12x_{2n+3} + 1500)$
   - $\frac{1}{25}(684y_{2n+3} - 276x_{2n+3} + 300)$
   - $\frac{1}{125}(684y_{2n+4} - 2748x_{2n+3} + 1500)$
   - $\frac{1}{125}(6732y_{2n+2} - 12x_{2n+4} + 14700)$
   - $\frac{1}{125}(6732y_{2n+3} - 276x_{2n+4} + 1500)$
   - $\frac{1}{25}(6732y_{2n+4} - 2748x_{2n+4} + 300)$
   - $\frac{1}{25}(138y_{2n+2} - 6y_{2n+3} + 300)$

- $\frac{1}{125}(687y_{2n+2}-3y_{2n+4}+1500)$
- $\frac{1}{25}(1374y_{2n+3}-138y_{2n+4}+300)$

3. Each of the following expressions is a cubical integer:
   - $\frac{1}{50}(3x_{3n+4}-19x_{3n+3}+9x_{n+2}-57x_{n+1})$
   - $\frac{1}{500}(3x_{3n+5}-187x_{3n+3}+9x_{n+3}-561x_{n+1})$
   - $\frac{1}{25}(18y_{3n+3}-2x_{3n+3}+54y_{n+1}-6x_{n+1})$
   - $\frac{1}{125}(18y_{3n+4}-46x_{3n+3}+54y_{n+2}-138x_{n+1})$
   - $\frac{1}{1225}(18y_{3n+5}-458x_{3n+3}+54y_{n+3}-1374x_{n+1})$
   - $\frac{1}{50}(19x_{3n+5}-187x_{3n+4}+57x_{n+3}-561x_{n+2})$
   - $\frac{1}{125}(114y_{3n+3}-2x_{3n+4}+342y_{n+1}-6x_{n+2})$
   - $\frac{1}{25}(114y_{3n+4}-46x_{3n+4}+342y_{n+2}-138x_{n+2})$
   - $\frac{1}{125}(114y_{3n+5}-458x_{3n+4}+342y_{n+3}-1374x_{n+2})$
   - $\frac{1}{1225}(1122y_{3n+3}-2x_{3n+5}+3366y_{n+1}-6x_{n+3})$
   - $\frac{1}{125}(1122y_{3n+4}-46x_{3n+5}+3366y_{n+2}-138x_{n+3})$
   - $\frac{1}{25}(1122y_{3n+5}-458x_{3n+5}+3366y_{n+3}-1374x_{n+3})$
   - $\frac{1}{25}(23y_{3n+3}-y_{3n+4}+69y_{n+1}-3y_{n+2})$
   - $\frac{1}{250}(229y_{3n+3}-y_{3n+5}+687y_{n+1}-3y_{n+3})$
   - $\frac{1}{50}(458y_{3n+4}-46y_{3n+5}+1374y_{n+2}-138y_{n+3})$

4. Each of the following expressions is a biquadratic integer:
   - $\frac{1}{50}(3x_{4n+5}-19x_{4n+4}+12x_{2n+3}-76x_{2n+2}+300)$
   - $\frac{1}{500}(3x_{4n+6}-187x_{4n+4}+12x_{2n+4}-748x_{2n+2}+3000)$
   - $\frac{1}{25}(18y_{4n+4}-2x_{4n+4}+72y_{2n+2}-8x_{2n+2}+150)$
   - $\frac{1}{125}(18y_{4n+5}-46x_{4n+4}+72y_{2n+3}-184x_{2n+2}+750)$
   - $\frac{1}{1225}(18y_{4n+6}-458x_{4n+4}+72y_{2n+4}-1832x_{2n+2}+7350)$

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- $\frac{1}{50}(19x_{4n+6} - 187x_{4n+5} + 76x_{2n+4} - 748x_{2n+3} + 300)$
- $\frac{1}{125}(114y_{4n+4} - 2x_{4n+5} + 456y_{2n+2} - 8x_{2n+3} + 750)$
- $\frac{1}{25}(114y_{4n+5} - 46x_{4n+5} + 456y_{2n+3} - 184x_{2n+3} + 150)$
- $\frac{1}{125}(114y_{4n+6} - 458x_{4n+5} + 456y_{2n+4} - 1832x_{2n+3} + 750)$
- $\frac{1}{1225}(1122y_{4n+4} - 2x_{4n+6} + 4488y_{2n+2} - 8x_{2n+4} + 7350)$
- $\frac{1}{125}(1122y_{4n+5} - 46x_{4n+6} + 4488y_{2n+3} - 184x_{2n+4} + 750)$
- $\frac{1}{25}(1122y_{4n+6} - 458x_{4n+6} + 4488y_{2n+4} - 1832x_{2n+4} + 150)$
- $\frac{1}{25}(23y_{4n+4} - y_{4n+5} + 92y_{2n+2} - 4y_{2n+3} + 150)$
- $\frac{1}{250}(229y_{4n+4} - y_{4n+6} + 916y_{2n+2} - 4y_{2n+4} + 1500)$
- $\frac{1}{50}(458y_{4n+5} - 46y_{4n+6} + 1832y_{2n+3} - 184y_{2n+4} + 300)$

5. Each of the following expressions is a quintic integer:
   - $\frac{1}{50}(3x_{5n+6} - 19x_{5n+5} + 15x_{3n+4} - 95x_{3n+3} + 6x_{n+2} - 38x_{n+1})$
   - $\frac{1}{500}(3x_{5n+7} - 187x_{5n+5} + 15x_{3n+5} - 935x_{3n+3} + 30x_{n+3} - 1870x_{n+1})$
   - $\frac{1}{25}(18y_{5n+5} - 2x_{5n+5} + 90y_{3n+3} - 10x_{3n+3} + 180y_{n+1} - 20x_{n+1})$
   - $\frac{1}{125}(18y_{5n+6} - 46x_{5n+5} + 90y_{3n+4} - 230x_{3n+3} + 180y_{n+2} - 460x_{n+1})$
   - $\frac{1}{1225}(18y_{5n+7} - 458x_{5n+5} + 90y_{3n+5} - 2290x_{3n+3} + 180y_{n+3} - 4580x_{n+1})$
   - $\frac{1}{50}(19x_{5n+7} - 187x_{5n+6} + 95x_{3n+5} - 935x_{3n+4} + 190x_{n+3} - 1870x_{n+2})$
   - $\frac{1}{125}(114y_{5n+5} - 2x_{5n+6} + 570y_{3n+3} - 10x_{3n+4} + 1140y_{n+1} - 20x_{n+2})$
   - $\frac{1}{25}(114y_{5n+6} - 46x_{5n+6} + 570y_{3n+4} - 230x_{3n+4} + 1140y_{n+2} - 460x_{n+2})$
   - $\frac{1}{125}(114y_{5n+7} - 458x_{5n+6} + 570y_{3n+5} - 2290x_{3n+4} + 1140y_{n+3} - 4580x_{n+2})$
   - $\frac{1}{1225}(1122y_{5n+5} - 2x_{5n+7} + 5610y_{3n+3} - 10x_{3n+5} + 11220y_{n+1} - 20x_{n+3})$
   - $\frac{1}{125}(1122y_{5n+6} - 46x_{5n+7} + 5610y_{3n+4} - 230x_{3n+5} + 11220y_{n+2} - 460x_{n+3})$
   - $\frac{1}{25}(1122y_{5n+7} - 458x_{5n+7} + 5610y_{3n+5} - 2290x_{3n+5} + 11220y_{n+3} - 4580x_{n+3})$
   - $\frac{1}{25}(23y_{5n+5} - y_{5n+6} + 115y_{3n+3} - 5y_{3n+4} + 230y_{n+1} - 10y_{n+2})$

- $\frac{1}{250}\left(229y_{5n+5}-y_{5n+7}+1145y_{3n+3}-5y_{3n+5}+2290y_{n+1}-10y_{n+3}\right)$
- $\frac{1}{50}\left(458y_{5n+6}-46y_{5n+7}+2290y_{3n+4}-230y_{3n+5}+4580y_{n+2}-460y_{n+3}\right)$

6. Relations among the solutions:
   - $10x_{n+2}=x_{n+3}+x_{n+1}$
   - $x_{n+2}=12y_{n+1}+5x_{n+1}$
   - $x_{n+3}=120y_{n+1}+49x_{n+1}$
   - $y_{n+2}=5y_{n+1}+2x_{n+1}$
   - $y_{n+3}=49y_{n+1}+20x_{n+1}$
   - $5x_{n+2}=12y_{n+2}+x_{n+1}$
   - $x_{n+3}=24y_{n+2}+x_{n+1}$
   - $5y_{n+3}=49y_{n+2}+2x_{n+1}$
   - $49x_{n+2}=12y_{n+3}+5x_{n+1}$
   - $49x_{n+3}=120y_{n+3}+x_{n+1}$
   - $5x_{n+3}=12y_{n+1}+49x_{n+2}$
   - $5y_{n+2}=y_{n+1}+2x_{n+2}$
   - $y_{n+3}=y_{n+1}+4x_{n+2}$
   - $x_{n+3}=12y_{n+2}+5x_{n+2}$
   - $y_{n+3}=5y_{n+2}+2x_{n+2}$
   - $5x_{n+3}=12y_{n+3}+x_{n+2}$
   - $49y_{n+2}=5y_{n+1}+2x_{n+3}$
   - $49y_{n+3}=y_{n+1}+20x_{n+3}$
   - $5y_{n+3}=y_{n+2}+2x_{n+3}$
   - $10y_{n+2}=y_{n+1}+y_{n+3}$

Remarkable observations

i. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2 below

Table: 2 Hyperbolas

S.NO	Hyperbola	(Y,X)
1	$3Y^2-2X^2=30000$	$(3x_{n+2}-19x_{n+1},23x_{n+1}-x_{n+2})$
2	$3Y^2-2X^2=3000000$	$(3x_{n+3}-187x_{n+1},229x_{n+1}-x_{n+3})$
3	$2Y^2-3X^2=45000$	$(54y_{n+1}-6x_{n+1},18x_{n+1}-12y_{n+1})$
4	$6Y^2-X^2=375000$	$(18y_{n+2}-46x_{n+1},114x_{n+1}-12y_{n+2})$
5	$6Y^2-X^2=36015000$	$(18y_{n+3}-458x_{n+1},1122x_{n+1}-12y_{n+3})$
6	$6Y^2-4X^2=60000$	$(19x_{n+3}-187x_{n+2},229x_{n+2}-23x_{n+3})$
7	$6Y^2-X^2=375000$	$(114y_{n+1}-2x_{n+2},18x_{n+2}-276y_{n+1})$
8	$6Y^2-X^2=15000$	$(114y_{n+2}-46x_{n+2},114x_{n+2}-276y_{n+2})$
9	$6Y^2-X^2=375000$	$(114y_{n+3}-458x_{n+2},1122x_{n+2}-276y_{n+3})$
10	$6Y^2-X^2=36015000$	$1122y_{n+1}-2x_{n+3},18x_{n+3}-2748y_{n+1}$
11	$6Y^2-X^2=375000$	$1122y_{n+2}-46x_{n+3},114x_{n+3}-2748y_{n+2}$

12	$6Y^2 - X^2 = 15000$	$(1122y_{n+3} - 458x_{n+3}, 1122x_{n+3} - 2748y_{n+3})$
13	$6Y^2 - X^2 = 15000$	$23y_{n+1} - y_{n+2}, 9y_{n+2} - 57y_{n+1}$
14	$6Y^2 - X^2 = 1500000$	$(229y_{n+1} - y_{n+3}, 9y_{n+3} - 561y_{n+1})$
15	$6Y^2 - X^2 = 60000$	$(458y_{n+2} - 46y_{n+3}, 114y_{n+3} - 1122y_{n+1})$

ii. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3 below

Table: 3 Parabolas

S.NO	Parabola	(Y, X)
1	$75Y - X^2 = 15000$	$(3x_{2n+3} - 19x_{2n+2} + 100, 23x_{n+1} - x_{n+2})$
2	$750Y - X^2 = 1500000$	$(3x_{2n+4} - 187x_{2n+2} + 1000, 229x_{n+1} - x_{n+3})$
3	$150Y - X^2 = 15000$	$(18y_{2n+2} - 2x_{2n+2} + 50, 18x_{n+1} - 12y_{n+1})$
4	$750Y - X^2 = 375000$	$(18y_{2n+3} - 46x_{2n+2} + 250, 114x_{n+1} - 12y_{n+2})$
5	$7350Y - X^2 = 36015000$	$(18y_{2n+4} - 458x_{2n+2} + 2450, 1122x_{n+1} - 12y_{n+3})$
6	$75Y - X^2 = 15000$	$(19x_{2n+4} - 187x_{2n+3} + 100, 229x_{n+2} - 23x_{n+3})$
7	$750Y - X^2 = 375000$	$(114y_{2n+2} - 2x_{2n+3} + 250, 18x_{n+2} - 276y_{n+1})$
8	$150Y - X^2 = 15000$	$(114y_{2n+3} - 46x_{2n+3} + 50, 114x_{n+2} - 276y_{n+2})$
9	$750Y - X^2 = 375000$	$(114y_{2n+4} - 458x_{2n+3} + 250, 1122x_{n+2} - 276y_{n+3})$
10	$7350Y - X^2 = 36015000$	$1122y_{2n+2} - 2x_{2n+4} + 2450, 18x_{n+3} - 2748y_{n+1}$
11	$750Y - X^2 = 375000$	$1122y_{2n+3} - 46x_{2n+4} + 250, 114x_{n+3} - 2748y_{n+2}$
12	$150Y - X^2 = 15000$	$(1122y_{2n+4} - 458x_{2n+4} + 50, 1122x_{n+3} - 2748y_{n+3})$
13	$150Y - X^2 = 15000$	$23y_{2n+2} - y_{2n+3} + 50, 9y_{n+2} - 57y_{n+1}$
14	$1500Y - X^2 = 1500000$	$(229y_{2n+2} - y_{2n+4} + 500, 9y_{n+3} - 561y_{n+1})$
15	$300Y - X^2 = 60000$	$(458y_{2n+3} - 46y_{2n+4} + 100, 114y_{n+3} - 1122y_{n+1})$

(i) Consider $m = x_{n+1} + y_{n+1}, n = x_{n+1}$, observe that $m > n > 0$. Treat m, n as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where $\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2$

Then the following interesting relations are observed.

(a) $\alpha - 3\beta + 2\gamma = 50$

(b) $\alpha + \beta - \gamma = \frac{4A}{P}$

(C) $3\left(\alpha - \frac{4a}{P}\right) = a$ nasty number.

3. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $x^2 = 6y^2 - 50$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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